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Improved Maximum Likelihood Estimation for the Akash Distribution

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ABSTRACT
We consider three analytical methods for reducing the finite-sample bias of the maximum likelihood estimator of the (scale) parameter in the Akash distribution. The latter distribution has flexible features that make it attractive for modelling lifetime data. Based on a simulation experiment, all three bias-reduction methods are found to be highly effective, and have the added merit of also reducing the mean squared error of the maximum likelihood estimator. The analytical results are also illustrated with six real-life data-sets.

KEYWORDS
Akash distribution, lifetime data, maximum likelihood estimation, bias reduction.

1. Introduction

Among the many statistical distributions that are used in reliability studies to analyze lifetime data, the Akash distribution proposed by Shanker [14] has certain advantages. It is simple, with just one (scale) parameter, but its density and hazard functions have more flexible shapes than those for competing distributions, such as the exponential and Lindley distributions. Notwithstanding its simplicity, maximum likelihood estimation of the scale parameter involves numerical optimization, and the resulting estimator is biased in finite samples.

In this paper we consider three analytical methods for reducing the order of magnitude of this bias. Specifically, we compare the "corrective" approach of Cox and Snell [5], the "preventive" method in Firth [6], and a more recent corrective method proposed in Godwin and Giles [9] that allows for less restrictive bias functions. These modified maximum likelihood estimators (MLE's) are compared in a Monte Carlo simulation experiment. The results show that as well as reducing bias, all three estimators also reduce the (percentage) mean squared error (MSE) of the original MLE. Of the three estimators, the Godwin-Giles estimator performs best in terms of bias reduction, but the other two estimators have a slight advantage in terms of MSE.

The Akash distribution and the analytical results relating to the various estimators are introduced in the next section. Section 3 outlines the simulation experiment and

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its associated results, and some illustrative applications with real data are provided in section 4. Some concluding remarks appear in section 5.

2. Theoretical results

If X follows the Akash distribution with a scale parameter, λ, its density function is

$$f(x) = [\lambda^3/(\lambda^2+2)](1+x^2)exp(-\lambda x) \quad ; \qquad x>0 \quad ; \qquad \lambda>0 \tag{1}$$

and its distribution function is

$$F(x) = 1 - [1+\lambda x(\lambda x+2)/(\lambda^2+2)]exp(-\lambda x). \tag{2}$$

Shanker (2015, p.67) shows that the r'th. moment of X about the origin is

$$\mu_r' = r![\lambda^2+(r+1)(r+2)]/[\lambda^r(\lambda^2+2)] \quad ; \quad r=1,2,... \tag{3}$$

implying that $E[X] = [\lambda^2+6]/[\lambda(\lambda^2+2)]$, and $Var.[X] = [\lambda^4+16\lambda^2+12]/[\lambda(\lambda^2+2)]^2$. This distribution has been generalized in various ways by authors such as Shanker and Shukla [16], Shanker et al. [17] and Abushal [1] However, the basic Akash distribution has considerable flexibility and is the focus of this paper.

Using a sample of n independent observations, with a sample mean of $\bar{x}$, the log-likelihood function based on equation 1 is

$$l = 3nlog(\lambda) - nlog(\lambda^2+2) - \lambda n\bar{x} + \sum_{i=1}^{n} log(1+x_i^2). \tag{4}$$

So,

$$\partial l/\partial\lambda = (3n/\lambda) - 2n\lambda/(\lambda^2+2) - n\bar{x}, \tag{5}$$

and the MLE($\tilde{\lambda}$) of λ is obtained by (numerically) solving the equation:

$$\bar{x}\lambda^3 - \lambda^2 + 2\bar{x}\lambda - 6 = 0 \tag{6}$$

for λ. Equating $\bar{x}$ and the above expression for $E[X]$, it follows immediately that the MLE of λ is also the method of moments estimator for that parameter.

The MLE for λ is consistent, asymptotically unbiased, and asymptotically efficient. However, its finite-sample properties have not been explored previously. In this paper we obtain an analytical approximation to the small-sample bias of this estimator, and derive three (approximately) "bias-corrected" estimators. The biases and MSEs of the latter estimators are compared with the corresponding properties of $\tilde{\lambda}$ itself in an extensive simulation experiment, described in section 3.

In what follows, we require the following results:

$$\partial^2 l/\partial\lambda^2 = -(3n/\lambda^2) + 4n\lambda^2/(\lambda^2+2)^2 - 2n/(\lambda^2+2) \tag{7}$$

$$\partial^3 l/\partial\lambda^3 = (6n/\lambda^3) - 16n\lambda^3/(\lambda^2+2)^3 + 12n\lambda/(\lambda^2+2)^2 \tag{8}$$

and we define:

$$\kappa_{11} = E[\partial^2 l/\partial\lambda^2] = \partial^2 l/\partial\lambda^2 \tag{9}$$

$$\kappa_{111} = E[\partial^3 l/\partial\lambda^3] = \partial^3 l/\partial\lambda^3 \tag{10}$$

and

$$\kappa_{11}^{(1)} = \partial\kappa_{11}/\partial\lambda = \partial^3 l/\partial\lambda^3 = \kappa_{111}. \tag{11}$$

The information measure is given by

$$K = -\kappa_{11} = (3n/\lambda^2) - 4n\lambda^2/(\lambda^2+2)^2 + 2n/(\lambda^2+2) \tag{12}$$

and so

$$\kappa^{11} = K^{-1} = [\lambda(\lambda^2+2)]^2/[3n(\lambda^2+2)^2 - 4n\lambda^4 + 2n\lambda^2(\lambda^2+2)]. \tag{13}$$

If we define

$$A = a_{11} = \kappa_{11}^{(1)} - 0.5\kappa_{111} = 0.5\kappa_{111} = (3n/\lambda^2) - 8n\lambda^3/(\lambda^2+2)^3 + 6n\lambda/(\lambda^2+2)^2, \tag{14}$$

then, following Coreiro and Klein [4], the bias of the MLE, $\tilde{\lambda}$ can be expressed as

$$B(\tilde{\lambda}) = (A/K^2) = \frac{n^{-1}\lambda(\lambda^2+2)[3(\lambda^2+2)^3 - 8\lambda^6 + 6\lambda^4(\lambda^2+2)]}{[3(\lambda^2+2)^2 - 4\lambda^4 + 2(\lambda^2+2)]^2} + O(n^{-2}). \tag{15}$$

The Cox-Snell bias-corrected estimator of λ is

$$\hat{\lambda} = \tilde{\lambda} - \tilde{B}(\tilde{\lambda}), \tag{16}$$

where $\tilde{B}(\tilde{\lambda}) = B(\tilde{\lambda})|_{\lambda=\tilde{\lambda}}$.

Firth's bias-corrected estimator, $\check{\lambda}$ is obtained by solving the equation,

$$\partial l/\partial\lambda - K(\lambda)B(\lambda) = 0 \tag{17}$$

for λ. The Cox-Snell and Firth estimators implicitly make the strong assumption that the bias function is "flat". Godwin and Giles [9] allow for a bias correction that avoids this restrictive assumption by proposing the estimator

$$\ddot{\lambda} = \tilde{\lambda} - \ddot{B}(\tilde{\lambda}), \tag{18}$$

where

$$\ddot{B}(\tilde{\lambda}) = B(\tilde{\lambda})|_{\lambda=\ddot{\lambda}}. \tag{19}$$

In equation 18, the bias function is evaluated at $\ddot{\lambda}$ (which is unbiased), rather than evaluated at $\tilde{\lambda}$ (which is biased). Of course, the difficulty with $\ddot{\lambda}$ is that there is no closed-form solution for this expression. Giles and Godwin rearrange equation 18, to get

$$\tilde{\lambda} = \ddot{\lambda} + \ddot{B}(\tilde{\lambda}), \tag{20}$$

and then the estimator $\ddot{\lambda}$ may then be obtained by substituting $[\lambda + B(\tilde{\lambda})]$ for λ in the log-likelihood function, equation 4, and maximizing the latter numerically.

Table 1. Summary results of a Monte Carlo study with 50,000 replications for six scale parameter values and five different sample sizes, $n = \{10, 15, 25, 50, 100\}$. We report the percentage biases of the maximum likelihood estimator, and the three "bias-corrected" estimators, of the scale parameter, λ.

n	$\%Bias(\tilde{\lambda})$	$\%Bias(\hat{\lambda})$	$\%Bias(\check{\lambda})$	$\%Bias(\ddot{\lambda})$
		$\lambda = 0.5$		
10	2.3869	-0.7314	-0.7321	-0.6438
15	1.5750	-0.4861	-0.4859	-0.4459
25	1.0335	-0.1963	-0.1955	-0.1864
50	0.4499	-0.1614	-0.1604	-0.1590
100	0.2066	-0.0983	-0.0973	-0.0977
		$\lambda = 1.0$		
10	2.9443	-0.5122	-0.4727	-0.3565
15	1.7167	-0.5306	-0.5147	-0.4638
25	1.1456	-0.1843	-0.1796	-0.1609
50	0.5275	-0.1290	-0.1290	-0.1218
100	0.2581	-0.0683	-0.0696	-0.0663
		$\lambda = 1.5$		
10	3.9390	-0.5226	-0.4255	-0.2159
15	2.5449	-0.3291	-0.2877	-0.1972
25	1.4393	-0.2397	-0.2252	-0.1934
50	0.6071	-0.2157	-0.2120	-0.2044
100	0.3230	-0.0853	-0.0842	-0.0811
		$\lambda = 2.0$		
10	5.2479	-0.4400	-0.2981	0.0537
15	3.2108	-0.4164	-0.3548	-0.2033
25	1.8957	-0.2152	-0.1932	-0.1399
50	0.8653	-0.1654	-0.1595	-0.1468
100	0.4714	-0.0390	-0.0368	-0.0354
		$\lambda = 2.5$		
10	6.4486	-0.3886	-0.2294	0.2769
15	4.0407	-0.3162	-0.2455	-0.0249
25	2.3295	-0.1978	-0.1725	-0.0941
50	1.1063	-0.1262	-0.1198	-0.1004
100	0.5684	-0.0408	-0.0392	-0.0353
		$\lambda = 3.0$		
10	7.4025	-0.3880	-0.2373	0.4096
15	4.6785	-0.2938	-0.2259	0.0589
25	2.7083	-0.1766	-0.1519	-0.0499
50	1.2988	-0.1080	-0.1018	-0.0764
100	0.6649	-0.0304	-0.0289	-0.0233

3. Simulation experiment

A Monte Carlo simulation experiment has been conducted to evaluate and compare the percentage biases and percentage MSEs of the original MLE ($\tilde{\lambda}$) and the three bias-corrected estimators ($\hat{\lambda}$, $\check{\lambda}$, and $\ddot{\lambda}$) for sample sizes ranging from $n = 10$ to $n = 100$, and values of λ between 0.5 and 3.0. These parameter values allow for a range of shapes for the Akash density, and reflect the estimated values obtained in a number of applications. For example, see Shanker [14] and Shanker and Fesshave [15].

Table 2. Summary results of a Monte Carlo study with 50,000 replications for six scale parameter values and five different sample sizes, $n = \{10, 15, 25, 50, 100\}$. We report the percentage mean squared errors of the maximum likelihood estimator, and the three "bias-corrected" estimators, of the scale parameter, λ.

n	$\%MSE(\tilde{\lambda})$	$\%MSE(\hat{\lambda})$	$\%MSE(\check{\lambda})$	$\%MSE(\ddot{\lambda})$
		$\lambda = 0.5$		
10	3.0247	2.8020	2.8032	2.8067
15	1.9291	1.8342	1.8345	1.8354
25	1.1380	1.1021	1.1021	1.1024
50	0.5439	0.5360	0.5360	0.5361
100	0.2662	0.2643	0.2643	0.2643
		$\lambda = 1.0$		
10	3.3418	2.9412	2.9504	2.9649
15	2.0442	1.8917	1.8941	1.8977
25	1.1793	1.1238	1.1244	1.1251
50	0.5641	0.5514	0.5515	0.5515
100	0.2726	0.2696	0.2695	0.2696
		$\lambda = 1.5$		
10	4.1763	3.4194	3.4370	3.4800
15	2.4733	2.1745	2.1793	2.1906
25	1.3695	1.2723	1.2732	1.2754
50	0.6347	0.6138	0.6138	0.6141
100	0.3103	0.3050	0.3050	0.3051
		$\lambda = 2.0$		
10	5.5470	4.2711	4.2909	4.3817
15	3.1275	2.6476	2.6534	2.6764
25	1.6654	1.5082	1.5095	1.5140
50	0.7622	0.7273	0.7274	0.7279
100	0.3709	0.3619	0.3620	0.3620
		$\lambda = 2.5$		
10	7.0442	5.2362	5.2469	5.3883
15	3.8170	3.1380	3.1419	3.1782
25	2.0100	1.7897	1.7907	1.7979
50	0.9101	0.8598	0.8599	0.8608
100	0.4382	0.4257	0.4257	0.4258
		$\lambda = 3.0$		
10	8.3419	6.0948	6.0948	6.2748
15	4.5845	3.7179	3.7179	3.7661
25	2.3293	2.0919	2.0521	2.0616
50	1.0494	0.9854	0.9855	0.9866
100	0.5035	0.4877	0.4877	0.4878

All of the simulations used $NREP = 50,000$ replications and were carried out using the R programming language (R Core Team, 2024). Random variates for the Akash distribution were generated by the acceptance-rejection method, using the R package ‘AcceptReject’ (Marinho [11]). The author’s R code (including that associated with the applications in section 4) can be downloaded from `https://github.com/DaveGiles1949/r-code`.

The simulated percentage biases and percentage MSEs of $\tilde{\lambda}$ are calculated as

$$\%Bias(\tilde{\lambda}) = 100[((1/NREP)\sum_{j=1}^{NREP}\tilde{\lambda}_j) - \lambda]/\lambda$$

and

$$\%MSE(\tilde{\lambda}) = 100[(1/NREP)\sum_{j=1}^{NREP}(\tilde{\lambda}_j - \lambda)^2]/\lambda^2$$

respectively, where $\tilde{\lambda}_j$ is the j’th. replication of the estimator, $\tilde{\lambda}$. The same calculations are made for the estimators, $\hat{\lambda}$, $\check{\lambda}$, and $\ddot{\lambda}$.

The simulation results are reported in Tables 1 and 2. In Table 1 we see that the (unadjusted) MLE of the scale parameter is biased upwards. In percentage terms, this bias (and the percentage MSE, shown in Table 2) increases with the true value of that parameter, for any given sample size. In all cases, the percentage biases and MSE’s in Tables 1 and 2 decrease as n increases, reflecting the consistency of all of the estimators. The Cox-Snell and Firth estimators reduce the (absolute) percentage bias of the original MLE by a similar substantial degree, and often by an order of magnitude when n is small. Often, this results in a small negative bias in estimation. In all cases, the reduction in bias is achieved with the added benefit of a decrease in the percentage MSE. The Godwin-Giles estimator generally has the smallest absolute percentage bias. It also has smaller percentage MSE than the original MLE, but generally its percentage MSE is slightly greater than those of the other two bias-corrected estimators, $\hat{\lambda}$ and $\check{\lambda}$.

4. Empirical applications

We present the results of six empirical applications to illustrate the consequences of applying the different bias-corrected MLEs of the Akash distribution’s scale parameter. The summary statistics for the data used in these applications, named “Data 1” to “Data 6” in Table 3, are reported in that table and can be downloaded from `https://github.com/DaveGiles1949/Data`.

Also shown in Table 3 are the values of the Anderson-Darling (A-D) statistic for testing the goodness-of-fit of the Akash distribution to each data-set. Here, the A-D test is one that has been modified using the “bias transformation” approach proposed by Raschke [13] for the beta and gamma distributions, and introduced by Giles [8] for the Akash distribution. The latter author shows that this A-D test out-performs other (modified) goodness-of-fit tests based on the empirical distribution function. The results in Table 3 suggest that the hypothesis that “Data 1” follows the Akash distribution should be rejected, at the 5% significance level; but this distribution is supported for the other samples at this level of significance.

“Data 1” is the sample used in the first application in Shanker [14], originally reported by Gross and Clark [10]. The data measure relief times (in minutes) of 20

Table 3. Results of six empirical applications. We report the summary statistics; the original and "bias-corrected" maximum likelihood estimates; and the modified Anderson-Darling "goodness-of-fit" test statistics. The 10% and 5% A-D critical values are 0.631 and 0.752. * and ** denote significance at the 10% and 5% levels respectively.

	Data 1	Data 2	Data 3	Data 4	Data 5	Data 6
n	20	31	34	20	11	70
mean	1.90	30.81	1.88	113.45	25.82	4921.00
median	1.70	29.90	1.15	119.00	26.00	4300.00
s.d.	0.70	7.25	1.95	35.79	13.58	2837.41
$\tilde{\lambda}$	1.1569	0.0970	1.1657	0.0264	0.1157	0.5919x10^{-3}
$\hat{\lambda}$	1.1369	0.0960	1.1538	0.0260	0.1122	0.5891x10^{-3}
$\check{\lambda}$	1.1371	0.0960	1.1539	0.0260	0.1122	0.6098x10^{-3}
$\ddot{\lambda}$	1.1374	0.0960	1.1540	0.0260	0.1123	0.5845x10^{-3}
A-D	0.8045**	0.4509	0.6388*	0.4297	0.2889	0.7383*

patients treated with an analgesic medication. "Data 2" is Shanker's second application data- set, and it measures the breaking strength of aircraft window glass. The sample of 31 observations is from Fuller et al. [7]. The "Data 3" sample comprises 34 observations on the amount of vinyl chloride found in clean upgradient monitoring wells, measured in mg/litre This data set was reported by Bhaumik et al. [2], and is Dataset 11 in the applications of the Akash distribution reported by Shanker and Fesshave [15]. "Data 4" was also used by Bhaumik et al. [2] (Table 4), and measures the survival time (in weeks) of 20 male mice that were exposed to gamma radiation. "Data 5" and "Data 6" are taken from the 'reliability' data set in the R package, 'survival' Therneau [19]. The first of these samples comprises 11 observations on the time of inspection for turbine wheels for cracks, in hundreds of hours. The second sample (with n =70) measures the time-to-failure of diesel generator fans (in hours).

The basic MLE ($\tilde{\lambda}$) and the three "bias-corrected" MLEs for the scale parameter of the Akash distribution are also reported in Table 3 for each application. In all cases, the bias-corrected estimates are essentially the same in value, and generally slightly smaller in value than the basic MLE. However, viewed in percentage terms, the differences between the value of $\tilde{\lambda}$ and the bias-corrected estimates are approximately -1.7%, -1.0%, -1.0%, -1.5%, and -3.0% for the first five application. In the case of "Data 6", this difference is between -1.2% and +3.0%, depending on the choice of bias correction

5. Conclusions

In this paper we have considered the finite-sample properties of the maximum likelihood estimator of the Akash distribution's sole parameter. This estimator is positively biased, and various ways of eliminating the first-order bias have been considered. The results of an extensive simulation experiment show that the Cox and Snell [5] "corrective" approach, and the Firth [6] "preventive" approach produce similar, and very successful, results in terms of reduced percentage bias and percentage mean squared error. The Godwin and Giles [9] approach, that allows for a less restrictive bias function, performs somewhat better than the other two approaches in terms of bias reduction, but at the expense of slightly less improvement in percentage mean squared error. Overall, the application of one or other of the bias corrections is recommended.

Several applications involving real-life data illustrate the extent to which each "bias correction" alters the numerical values of the estimates of the Akash distribution's scale parameter in relatively small samples. Decreases in the estimates' values by an order of 1% to 3% is found to be typical.

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